KIT Department of Informatics
Institute for Anthropomatics and Robotics (IAR)
High Performance Humanoid Technologies (H^{2}T)

Robotics I, WS 2024/2025

Solution Sheet 2

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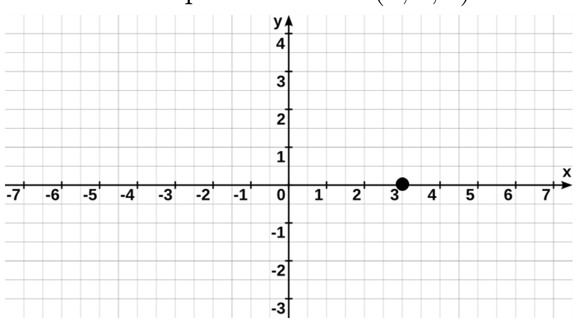
Solution 1 (Vorwärtskinematik, Denavit-Hartenberg)

1. **Interpretation of a Pose**

   i.) ${}^{BCS}T_{root}$ describes a rotation around the z axis by 30° and a translation by 3 in x-direction.

   ii.) The robot is positioned at $(3, 0, 0)^\top$.



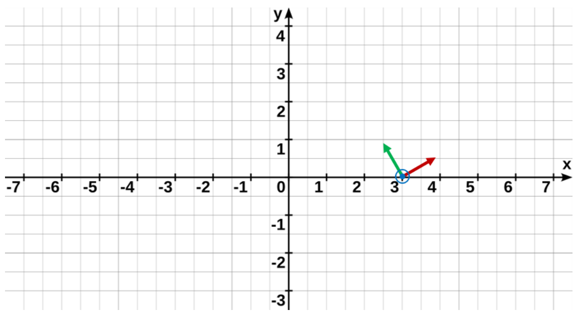
iii.) $\mathbf{v}_{x,root} = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\mathbf{v}_{y,root} = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $\mathbf{v}_{z,root} = \begin{pmatrix}0\\0\\1\end{pmatrix}$, $\mathbf{v}_{o,root} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

iv.) $\mathbf{v}_{BCS} = ^{BCS} T_{root} \cdot \mathbf{v}_{root}$, specifically:

$$\begin{pmatrix}\frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 & 3\\ \frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \cdot \begin{pmatrix}1\\0\\0\\1\end{pmatrix} = \begin{pmatrix}3+\frac{1}{2}\sqrt{3}\\ \frac{1}{2}\\ 0\\ 1\end{pmatrix} \Rightarrow \mathbf{v}_{x,BCS} = \begin{pmatrix}3+\frac{1}{2}\sqrt{3}\\ \frac{1}{2}\\ 0\end{pmatrix}$$

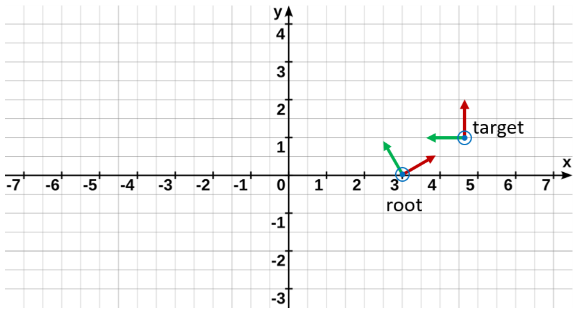
Similarly:

$$\mathbf{v}_{y,BCS} = \begin{pmatrix}\frac{5}{2}\\ \frac{1}{2}\sqrt{3}\\ 0\end{pmatrix} \; \mathbf{v}_{z,BCS} = \begin{pmatrix}3\\0\\1\end{pmatrix} \; \mathbf{v}_{o,BCS} = \begin{pmatrix}3\\0\\0\end{pmatrix}$$



2. **Conversion Between Coordinate Systems**

i.)



ii.) $^{BCS}T_{root} \cdot ^{root} T_{target} = ^{BCS} T_{target}$

$\Rightarrow \quad ^{root}T_{target} = \left(^{BCS}T_{root}\right)^{-1} \cdot ^{BCS} T_{target}$

$$\left(^{BCS}T_{root}\right)^{-1} = \begin{pmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 & 3 \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2}\sqrt{3} & \frac{1}{2} & 0 & -\frac{3}{2}\sqrt{3} \\ -\frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 & \frac{3}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$^{root}T_{target} = \begin{pmatrix} \frac{1}{2}\sqrt{3} & \frac{1}{2} & 0 & -\frac{3}{2}\sqrt{3} \\ -\frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 & \frac{3}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 & 0 & 3+\sqrt{3} \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} & 0 & 2 \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

iii.) $^{BCS}T_{target} \cdot ^{target} T_{root} = ^{BCS} T_{root}$

$\Rightarrow \quad ^{target}T_{root} = \left(^{BCS}T_{target}\right)^{-1} \cdot ^{BCS} T_{root}$

$$\left(^{BCS}T_{target}\right)^{-1} = \begin{pmatrix} 0 & -1 & 0 & 3+\sqrt{3} \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & -3-\sqrt{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$^{target}T_{root} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & -3-\sqrt{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 & 3 \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 & -1 \\ -\frac{1}{2}\sqrt{3} & \frac{1}{2} & 0 & \sqrt{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

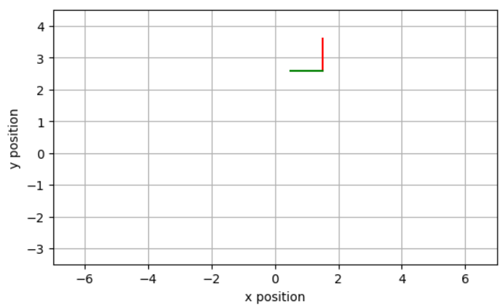
3. **Local and Global Transformation**

i.)

$$T = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} & 0 & -1 \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} & 0 & \sqrt{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

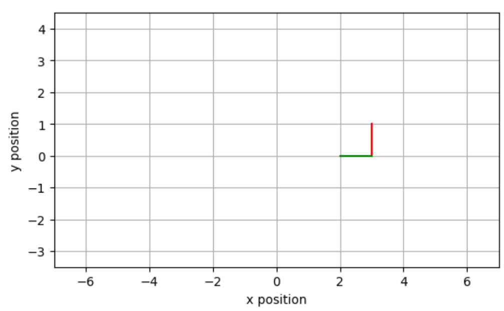
ii.)

$$^{BCS}T_{trans,L} = T \cdot ^{BCS} T_{root} = \begin{pmatrix} 0 & -1 & 0 & \frac{3}{2} \\ 1 & 0 & 0 & \frac{3}{2}\sqrt{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



iii.)

$$^{BCS}T_{trans,R} = ^{BCS} T_{root} \cdot T = \begin{pmatrix} 0 & -1 & 0 & 3 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

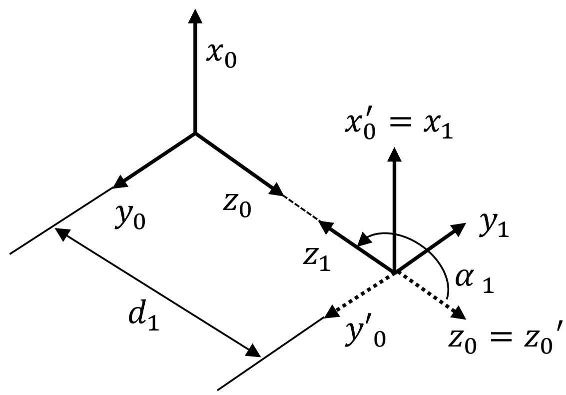


iv.) From the left: global transformation, from the right: local transformation

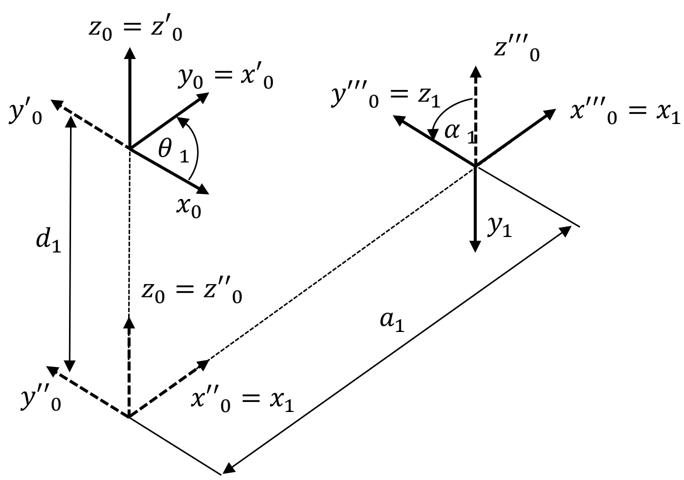
Solution 2 (Vorwärtskinematik, Denavit-Hartenberg)

1. Transformations of coordinate systems

   i.) $\theta_1 = 0°$, $d_1 = 60\,\text{mm}$, $a_1 = 0\,\text{mm}$, $\alpha_1 = 180°$



   ii.) $\theta_1 = 90°$, $d_1 = -30\,\text{mm}$, $a_1 = 60\,\text{mm}$, $\alpha_1 = -90°$



2. DH parameters

Joint	$\boldsymbol{\theta}$	$\boldsymbol{d}$	$\boldsymbol{a}$	$\boldsymbol{\alpha}$
J1	$90° + \theta_1$	$500\,\text{mm}$	$0\,\text{mm}$	$90°$
J2	$90° + \theta_2$	$200\,\text{mm}$	$0\,\text{mm}$	$90°$
J3	$0°$	$250\,\text{mm} + d_3$	$0\,\text{mm}$	$0°$

Solution 3 4

1. DH Parameters of the Crane

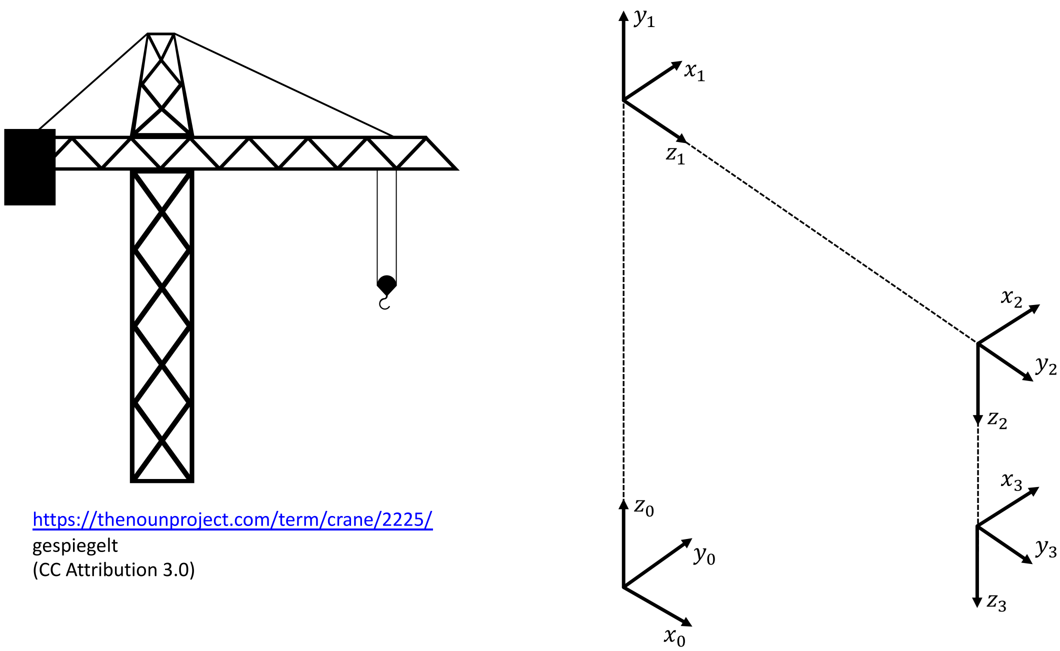


Figure 1: Structure of the crane

Joint	$\boldsymbol{\theta}$	$\boldsymbol{d}$	$\boldsymbol{a}$	$\boldsymbol{\alpha}$
Joint 1	$\theta_1 + 90°$	20	0	90°
Joint 2	0°	$2 \leq d_2 \leq 15$	0	90°
Joint 3	0°	$0 \leq d_3 \leq 20$	0	0°

Transformation matrix of the end effector

$$T_{0,1} = \begin{pmatrix} \cos(\theta_1 + 90^\circ) & 0 & \sin(\theta_1 + 90^\circ) & 0 \\ \sin(\theta_1 + 90^\circ) & 0 & -\cos(\theta_1 + 90^\circ) & 0 \\ 0 & 1 & 0 & 20 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_{1,2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_{2,3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} T_{0,3} &= T_{0,1}T_{1,2}T_{2,3} \\ &= \begin{pmatrix} \cos(\theta_1 + 90^\circ) & \sin(\theta_1 + 90^\circ) & 0 & \sin(\theta_1 + 90^\circ)d_2 \\ \sin(\theta_1 + 90^\circ) & -\cos(\theta_1 + 90^\circ) & 0 & -\cos(\theta_1 + 90^\circ)d_2 \\ 0 & 0 & -1 & 20 \\ 0 & 0 & 0 & 1 \end{pmatrix} T_{2,3} \\ &= \begin{pmatrix} \cos(\theta_1 + 90^\circ) & \sin(\theta_1 + 90^\circ) & 0 & \sin(\theta_1 + 90^\circ)d_2 \\ \sin(\theta_1 + 90^\circ) & -\cos(\theta_1 + 90^\circ) & 0 & -\cos(\theta_1 + 90^\circ)d_2 \\ 0 & 0 & -1 & -d_3 + 20 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

2. General Jacobian matrix of the end effector

$$
\begin{aligned}
x &= \sin(\theta_1 + 90°)d_2 \\
y &= -\cos(\theta_1 + 90°)d_2 \\
z &= -d_3 + 20 \\
\beta &= \operatorname{asin}(-n_z) = \operatorname{asin}(-0) = 0 \\
\cos(\beta)\sin(\alpha) &= o_z = 0,\ \cos(\beta)\cos(\alpha) = a_z = -1 \Rightarrow \alpha = \pi \\
\gamma &= \operatorname{atan}\left(\frac{n_y}{n_x}\right) = \operatorname{atan}\left(\frac{\sin(\theta_1 + 90°)}{\cos(\theta_1 + 90°)}\right) = \operatorname{atan}(\tan(\theta_1 + 90°)) = \theta_1 + 90°
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial x}{\partial \theta_1} &= \cos(\theta_1 + 90°)d_2 \\
\frac{\partial y}{\partial \theta_1} &= \sin(\theta_1 + 90°)d_2 \\
\frac{\partial z}{\partial \theta_1} &= 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial x}{\partial d_2} &= \sin(\theta_1 + 90°) \\
\frac{\partial y}{\partial d_2} &= -\cos(\theta_1 + 90°) \\
\frac{\partial z}{\partial d_2} &= 0
\end{aligned}
$$

x and y do not depend on d_3, so that its derivation with respect to d_3 is 0. The derivation of z with respect to d_3 is -1.

α, β and γ do not depend on d_2 and d_3, so that the respective derivations are 0. Only γ depends on θ_1, with the partial derivation with respect to θ_1 being 1. The derivations with respect to α and β are again 0.

Thus, the resulting Jacobian matrix J is:

$$
J = \begin{pmatrix}
\cos(\theta_1 + 90°)d_2 & \sin(\theta_1 + 90°) & 0 \\
\sin(\theta_1 + 90°)d_2 & -\cos(\theta_1 + 90°) & 0 \\
0 & 0 & -1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix}
$$

To get the Jacobian matrix for specific end effector configurations, θ_1, d_2, d_3 need to be substituted in J by their respective values.

3. Calculation of the end effector velocity

i.)

$$J(\mathbf{q}_1)\mathbf{p}_1 = \begin{pmatrix} -10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -10 \\ 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

ii.)

$$J(\mathbf{q}_1)\mathbf{p}_2 = \begin{pmatrix} -10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 10 \\ -1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

iii.)

$$J(\mathbf{q}_2)\mathbf{p}_2 = \begin{pmatrix} 0 & -1 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

iv.)

$$J(\mathbf{q}_2)\mathbf{p}_3 = \begin{pmatrix} 0 & -1 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ -2 \\ 0 \\ 0 \\ 2 \end{pmatrix}$$

Alternatively, the Jacobian matrix for a specific configuration can be determined without previously calculating the general Jacobian matrix. The i-th column of the Jacobian matrix is calculated as follows, as known from the lecture:

i.) Translation along axis $\mathbf{v}_i$: $\frac{\partial f(\mathbf{q})}{\partial \mathbf{q}_i} = \begin{pmatrix} \mathbf{v}_i \\ \mathbf{0} \end{pmatrix} \in \mathbb{R}^6$

ii.) Rotation around axis $\mathbf{v}_i$, with the joint positioned at $\mathbf{r}_i$:
$\frac{\partial f(\mathbf{q})}{\partial \mathbf{q}_i} = \begin{pmatrix} \mathbf{v}_i \times (f_{position}(\mathbf{q}) - \mathbf{r}_i) \\ \mathbf{v}_i \end{pmatrix} \in \mathbb{R}^6$

Let $f(\mathbf{q})$ be the function describing the pose of the end effector in the basis coordinate system. Example for the calculation of the Jacobian matrix for the configuration $\mathbf{q}_1$:

- Joint 1 rotates around the z-axis that is located at the origin:
$$\frac{\partial f(\mathbf{q}_1)}{\partial \mathbf{q}_{1,1}} = \begin{pmatrix} \mathbf{e}_z \times (f(\mathbf{q}_1) - \mathbf{0}) \\ \mathbf{e}_z \end{pmatrix} = \begin{pmatrix} \mathbf{e}_z \times \left(\begin{pmatrix} 0 \\ 10 \\ 10 \end{pmatrix} - \mathbf{0} \right) \\ \mathbf{e}_z \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} -10 \\ 0 \\ 0 \end{pmatrix} \\ \mathbf{e}_z \end{pmatrix}$$

- Joint 2 gets moved along the y-axis, as the crane is rotated by 90°:
$$\frac{\partial f(\mathbf{q}_1)}{\partial \mathbf{q}_{1,2}} = \begin{pmatrix} \mathbf{e}_y \\ \mathbf{0} \end{pmatrix}$$

- Joint 3 moves along the negative z-axis, as the zero position of the joint is at +20 m:
$$\frac{\partial f(\mathbf{q}_1)}{\partial \mathbf{q}_{1,3}} = \begin{pmatrix} -\mathbf{e}_z \\ \mathbf{0} \end{pmatrix}$$

Using the results for joint 1, joint 2 and joint 3 as columns of the Jacobian matrix, this results in:

$$J(\mathbf{q}_1) = \begin{pmatrix} -10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$